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High-order behaviour of the critical exponents for the scalar ϕ^3 model with imaginary coupling

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Abstract. For real coupling constants the vertex functions for the ϕ^3 model have a real and imaginary part. We present a method of renormalising both parts of the vertex functions for small coupling g_0. A consequence of our method is that the renormalisation group functions develop a non-perturbative imaginary part. This leads to an imaginary part for the critical exponents valid for $6-d=\varepsilon<0$, where d is the dimension of the space. A dispersion relation in ε then yields the high-order behaviour of the critical exponents as a power series in ε. In particular we obtain the values of the overall universal coefficients, c, of the factorial growth.

1. Introduction

In this paper we are interested in studying the following model,

$$\mathscr{H}=\int \mathrm{d}^d\boldsymbol{x}\,[\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_0^2\phi^2+1/3!g_0\phi^3+1/4!v_0\phi^4], \tag{1.1}$$

for imaginary g_0. This model was considered by Fisher (1978) in his study of Yang–Lee edge singularities. The exponent σ which characterises the Yang–Lee edge singularity is related to the value of the critical exponent η for the above Hamiltonian by

$$\sigma=(d-2+\eta)/(d+2-\eta). \tag{1.2}$$

It has been shown (Kirkham and Wallace 1978) that the Hamiltonian is stable in the absence of the ϕ^4 interaction and that the ϕ^4 interaction is an irrelevant operator. As a result of this, the ϕ^4 interaction can be neglected in any calculation of the critical exponents and the critical exponents have an oscillatory nature in the ε expansion ($\varepsilon=6-d$ in this case). Both these facts are a direct consequence of g_0 being imaginary.

The ε expansion is asymptotic and therefore before one can obtain accurate values for the exponents the high-order behaviour of the expansion has to be calculated. For a model in which the ε expansion is oscillatory, the high-order behaviour of a given exponent is typically of the form

$$c(-1)^k a^k k^b k!\,\varepsilon^k(1+\mathrm{O}(k^{-1})). \tag{1.3}$$

The various values of a and b have previously been calculated for this model (Kirkham and Wallace 1978). Using these values, resummation techniques have been employed by de Alcantara Bonfim *et al* (1981) to obtain η and σ. The methods of resummation used in their paper did not require any knowledge of c, although resummation schemes using the value of c have been discussed by Le Guillou and Zinn-Justin (1980).

For real values of g_0 there is no stable ground state and as a consequence of this the vertex functions become complex. The imaginary part can be calculated by studying the classical solutions (instantons) to the field equations of the theory (Zinn-Justin 1982). The real part is just the usual perturbative expression apart from some exponentially small terms which can be neglected for the purposes of this calculation. Once the bare vertex functions have been calculated the expressions have to be renormalised in such a way that they have a finite limit as $\varepsilon \to 0$. We present a renormalisation scheme which is a straightforward adaptation of the method discussed by McKane and Wallace (1983) for the ϕ^4 model with negative coupling. Indeed, given the lack of understanding of the systematics in their work, the major motivation for this calculation was to see if their method of renormalising and of obtaining the high-order behaviour of the critical exponents is more generally applicable.

The theory is renormalised by using an extended minimal subtraction scheme, in which the renormalisation group functions have an imaginary non-perturbative part. Analytically continuing these functions to $\varepsilon < 0$, we find a fixed point, and the values of the critical exponents at this fixed point have an imaginary part. A dispersion relation in ε then gives the high-order behaviour of the critical exponents for the Hamiltonian given by equation (1), and in particular we obtain the values of the overall universal coefficients c.

The renormalisation scheme is discussed in detail in § 2. In § 3 the renormalisation group functions are calculated and using these the imaginary parts of the critical exponents valid for $\varepsilon < 0$ are evaluated. Section 4 is devoted to calculating the high-order behaviour of the exponents valid for $\varepsilon > 0$.

2. Renormalisation

The definition of the vertex function used here is the one described by Amit (1978). The imaginary and real parts to the vertex functions are evaluated using dimensional regularisation for $m_0^2 = 0$. The imaginary part for ϕ^3 models with real g_0 has been calculated in McKane (1979), by expanding about the instanton solution of six dimensions. However, it was pointed out in McKane and Wallace (1983) that the arguments of the extended minimal subtraction scheme go through more cleanly if the expansion is performed about a slightly modified instanton (Drummond and Shore 1979). The analogous instanton for the ϕ^3 theory is

$$\phi_c = -48\lambda^2/g_0[1+\lambda^2(x-x_0)^2]^{d/2-1} \tag{2.1}$$

and the only effect of expanding about this instanton is to change the function $\tilde{\phi}(q)$ in equation (43) of McKane (1979) to its value at $d = 6$. Bearing in mind this change and the fact that we are employing different normalisations here, the imaginary part of the vertex functions can be obtained from equation (44) of McKane (1979),

$$\operatorname*{Im}_{\arg g_0 = 0}\Gamma^{(N)}(q_i) = -c_b \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{d-N(d-2)/2} \left(\frac{A\lambda^\varepsilon}{k_d g_0^2}\right)^{(d+1+N)/2}$$
$$\times \exp\left(-\frac{A\lambda^\varepsilon}{k_d g_0^2}\right) \prod_{i=1}^{N} \frac{q_i^2}{\lambda^2} \tilde{\phi}(q_i/\lambda)(1+O(\varepsilon, g_0^2)) \tag{2.2}$$

where

$$\tilde{\phi}(q) = -5^{1/2}3^{1/2}\pi^{3/2}2^2 q^{-1}K_1(q),$$

$$A = \tfrac{12}{5}(1+\varepsilon/4+\varepsilon\ln 2-\varepsilon\gamma+O(\varepsilon^2)),$$

$$c_b = 2^{5/2}\pi^{-7/2}(2\pi)^{-9/10}\exp\left(\frac{18}{5\varepsilon}+\frac{11\zeta'(2)}{2}-\frac{3\zeta'(4)}{2\pi^4}-\frac{9\gamma}{2}-\frac{527}{144}\right).$$

In (2.2), $k_d = S_d/(2\pi)^d$, ζ' are derivatives of the Riemann zeta function, γ is Euler's constant and K_1 is a modified Bessel function. For arg $g_0 = -0$ the imaginary part has exactly the same form apart from the fact that there is no overall minus sign.

In (2.2) the integration over the parameter λ is a consequence of using collective coordinates in the instanton calculation, and as we shall see below it gives rise to some poles in ε. The modified Bessel function K_1 has an exponential decay for large argument, and because of this the integral converges for small λ regardless of the value of ε. For the large λ we can show that

$$(q_i^2/\lambda^2)\tilde{\phi}(q_i/\lambda) = -5^{1/2}3^{1/2}\pi^{3/2}2^2[(1+(q^2/2\lambda^2)(\ln(q_i/2\lambda)+\gamma-\tfrac{1}{2})+O(\lambda^{-3})] \tag{2.3}$$

and so using this expansion we can see tnat the integral diverges for large λ at $\varepsilon = 0$ in $\Gamma^{(3)}$ and $\Gamma^{(2)}$ only.

We will discuss in detail the renormaliation of the theory for arg $g_0 = 0$. However, for arg $g_0 = -0$ the theory can be renormalised in a similar fashion.

Let us first look at $\Gamma^{(3)}$. To extract the pole term we introduce an arbitrary non-exceptional momentum scale μ as a lower cut-off in the λ integral. The integral from 0 to μ will give a finite contribution at $\varepsilon = 0$ and we have not made this term explicit. From (2.2) and (2.3) we have

$$\underset{\arg g_0=0}{\mathrm{Im}\Gamma^{(3)}(\boldsymbol{q}_i)} = -c_b\int_\mu^\infty \frac{d\lambda}{\lambda}\lambda^3\left(\frac{A\lambda^\varepsilon}{k_d g_0^2}\right)^{(d+4)/2}\exp\left(-\frac{A\lambda^\varepsilon}{k_d g_0^2}\right)$$
$$\times(-1)(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3(1+O(\lambda^{-1}))(1+O(\varepsilon, g_0^2)). \tag{2.4}$$

The $O(\lambda^{-1})$ terms give a finite result at $\varepsilon = 0$ and so we are only interested in the first term. Evaluating the remaining integral for $\varepsilon > 0$, we obtain

$$\underset{\arg g_0=0}{\mathrm{Im}\Gamma^{(3)}(\boldsymbol{q}_i)} = \frac{c_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3\mu^{\varepsilon/2}\left(\frac{A\mu^\varepsilon}{k_d g_0^2}\right)^{(d+2)/2}\exp\left(-\frac{A\mu^\varepsilon}{k_d g_0^2}\right)(1+O(\varepsilon, g_0^2)). \tag{2.5}$$

If we now do the usual perturbative renormalisations (de Alcantara Bonfim *et al* 1981)

$$Z_\phi^p = 1 - k_d g^2/6\varepsilon + O(g^4), \tag{2.6}$$

$$\mu^{-\varepsilon/2}g_0 = g - 3k_d g^3/4\varepsilon + O(g^5), \tag{2.7}$$

where we have not made the higher-order terms explicit, then this guarantees that the real part is finite at $\varepsilon = 0$. The effect of these renormalisations on the imaginary part is that Im $\Gamma^{(3)}$ becomes

$$\underset{\arg g=0}{\mathrm{Im}\Gamma^{(3)}(\boldsymbol{q}_i)} = \frac{\tilde{c}_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3\mu^{\varepsilon/2}\left(\frac{A}{k_d g^2}\right)^{(d+2)/2}\exp\left(-\frac{A}{k_d g^2}\right)(1+O(\varepsilon, g^2)) \tag{2.8}$$

where $\tilde{c}_b = c_b\exp[-\tfrac{18}{5}\varepsilon(1+\varepsilon/4+\varepsilon\ln 2-\varepsilon\gamma))]$, and hence $\tilde{c}_b$ is finite at $\varepsilon = 0$. We are therefore just left with a simple pole which we remove as follows. The full expression for $\Gamma^{(3)}$ is

$$\Gamma^{(3)}_{\arg g=0}(\boldsymbol{q}_i)\mu^{-\varepsilon/2} = g + O(g^3) + \frac{i\tilde{c}_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3\left(\frac{A}{k_d g^2}\right)^{(d+2)/2} \times \exp\left(-\frac{A}{k_d g^2}\right)(1+O(\varepsilon, g^2)) \tag{2.9}$$

and so we now define a full renormalised coupling constant

$$g_R = g + \frac{i\tilde{c}_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3\left(\frac{A}{k_d g^2}\right)^{(d+2)/2}\exp\left(-\frac{A}{k_d g^2}\right)(1+O(g^2)). \tag{2.10}$$

Inverting this we have

$$g = g_R - \frac{i\tilde{c}_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3\left(\frac{A}{k_d g_R^2}\right)^{(d+2)/2}\exp\left(-\frac{A}{k_d g_R^2}\right)(1+O(g_R^2)) \tag{2.11}$$

and this extra renormalisation renders $\Gamma^{(3)}$ finite. As we shall see next, we will need an extra wavefunction renormalisation, but it is of higher order in g_R^2 and so it does not affect $\Gamma^{(3)}$.

Now let us turn our attention to $\Gamma^{(2)}$. We again use μ as a lower cut-off for the λ integral. From (2.2) and (2.3) we have

$$\operatorname{Im} \Gamma^{(2)}_{\arg g_0=0}(\boldsymbol{q}) = -c_b\int_\mu^\infty \frac{d\lambda}{\lambda}\lambda^2\left(\frac{A\lambda^\varepsilon}{k_d g_0^2}\right)^{(d+3)/2}\exp\left(-\frac{A\lambda^\varepsilon}{k_d g_0^2}\right)(5^{1/2}3^{1/2}\pi^{3/2}2^2)^2 \times[1+(q^2/\lambda^2)(\gamma-\tfrac{1}{2}+\ln(q/2\lambda))+O(\lambda^{-4})](1+O(\varepsilon, g_0^2)). \tag{2.12}$$

The $O(\lambda^{-4})$ terms give a finite result at $\varepsilon=0$. However, the remaining terms will give rise to poles and in particular we have the possibility of a momentum dependent pole. The first term is quadratically divergent and independent of q—it can therefore be removed by a mass counterterm. The exact form of this counterterm does not matter since it has no effect on the renormalisation group functions.

Evaluating the remaining integrals for $\varepsilon>0$, we have, putting in the real part,

$$\Gamma^{(2)}_{\arg g_0=0}(\boldsymbol{q}) = q^2\left[1+\frac{1}{6}q^{-\varepsilon}g_0^2k_d\left(\frac{1}{\varepsilon}+\frac{7}{12}+O(\varepsilon)\right)+O(g_0^4)\right] - \frac{i\tilde{c}_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^2q^2\left(\frac{A\mu^\varepsilon}{k_d g_0^2}\right)^{(d+1)/2}(\gamma-\tfrac{1}{2}+\ln(q/2\mu)) \times\exp\left(-\frac{A\mu^\varepsilon}{k_d g_0^2}\right)(1+O(\varepsilon, g_0^2)). \tag{2.13}$$

If we now do the perturbative wavefunction renormalisation (2.6) and the perturbative coupling constant renormalisation (2.7) then we are left with

$$\Gamma^{(2)}_{\arg g=0}(\boldsymbol{q}) = q^2[1+\tfrac{1}{6}k_d g^2(\tfrac{7}{12}+\ln(\mu/q)+O(\varepsilon))+O(g^4)] - \frac{i\tilde{c}_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^2q^2\left(\frac{A}{k_d g^2}\right)^{(d+1)/2}(\gamma-\tfrac{1}{2}+\ln(q/2\mu)) \times\exp\left(-\frac{A}{k_d g^2}\right)(1+O(\varepsilon, g^2)). \tag{2.14}$$

Substituting (2.11) into this, it turns out that the ln q pole cancels and

$$\Gamma^{(2)}(\boldsymbol{q}) = q^2[1+\tfrac{1}{6}k_d g_R^2(\tfrac{7}{12}+\ln(\mu/q)+O(\varepsilon))+O(g_R^4)]$$
$$-\frac{\mathrm{i}\tilde{c}_b q^2}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^2\left(\frac{A}{k_d g_R^2}\right)^{(d+1)/2}(\gamma-\tfrac{1}{2}-\ln 2+\tfrac{7}{12})$$
$$\times\exp\left(-\frac{A}{k_d g_R^2}\right)(1+O(\varepsilon, g_R^2)). \tag{2.15}$$

This expression can be rendered finite by doing an extra non-perturbative wavefunction renormalisation given by

$$Z_\phi^{\mathrm{np}} = 1+\frac{\mathrm{i}\tilde{c}_b}{\varepsilon}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^2\left(\frac{A}{k_d g_R^2}\right)^{(d+1)/2}(\gamma-\tfrac{1}{2}-\ln 2+\tfrac{7}{12})$$
$$\times\exp\left(-\frac{A}{k_d g_R^2}\right)(1+O(g_R^2)). \tag{2.16}$$

Finally we look at the renormalisation of the vertex functions with insertions of $\frac{1}{2}\phi^2$ operators. The imaginary part can be calculated using the results in McKane (1979) and equation (2.1),

$$\operatorname*{Im}_{\arg g_0=0}\Gamma^{(N,M)}(\boldsymbol{q}_i;\boldsymbol{p}_i)$$
$$= c_b\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\lambda^{[d-N(d-2)]/2-2M}\left(\frac{A\lambda^\varepsilon}{k_d g_0^2}\right)^{(d+1+N+2M)/2}$$
$$\times\exp\left(-\frac{A\lambda^\varepsilon}{k_d g_0^2}\right)\prod_{i=1}^{N}\frac{q_i^2}{\lambda^2}\tilde{\phi}(q_i/\lambda)\prod_{i=1}^{M}\tilde{\psi}(p_i/\lambda)(1+O(\varepsilon, g_0^2)) \tag{2.17}$$

where

$$\tilde{\psi}(p) = 5\times 2^{-2}p^{1-\varepsilon/2}K_{-1+\varepsilon/2}(p).$$

The asymptotic expansion of $\tilde{\psi}(p/\lambda)$ for large λ is

$$\tilde{\psi}(p/\lambda) = 5\times 2^{-2}(1+O(\varepsilon))(1+O(\lambda^{-2})) \tag{2.18}$$

and so we can see that the λ integral diverges for $\varepsilon=0$ in $\Gamma^{(2,1)}$ only. To extract the pole we proceed analogously to $\Gamma^{(2)}$ and $\Gamma^{(3)}$, and we obtain

$$\Gamma^{(2,1)}_{\arg g_0=0}(\boldsymbol{q}_i;\boldsymbol{p}) = 1+p^{-\varepsilon}k_d g_0^2\left(\frac{1}{\varepsilon}+O(1)\right)+O(g_0^4)$$
$$+\frac{\mathrm{i}c_b}{\varepsilon}(5^2 3\pi^3 2^2)\left(\frac{A\mu^\varepsilon}{k_d g_0^2}\right)^{(d+3)/2}\exp\left(-\frac{A\mu^\varepsilon}{k_d g_0^2}\right)(1+O(\varepsilon, g_0^2)). \tag{2.19}$$

The normal perturbative $Z^{\mathrm{p}}_{\phi^2}$ renormalisation is (de Alcantara Bonfim *et al* 1981)

$$Z^{\mathrm{p}}_{\phi^2} = 1-k_d g^2/\varepsilon+O(g^4) \tag{2.20}$$

and so using (2.7) and (2.20)

$$\Gamma^{(2,1)}_{\arg g=0}(\boldsymbol{q}_i;\boldsymbol{p}) = 1+O(g^2)+\frac{\mathrm{i}\tilde{c}_b}{\varepsilon}(5^2 3\pi^3 2^2)\left(\frac{A}{k_d g^2}\right)^{(d+3)/2}$$
$$\times\exp\left(-\frac{A}{k_d g^2}\right)(1+O(\varepsilon, g^2)). \tag{2.21}$$

Substituting (2.11) into this, the remaining expression can be rendered finite by an additional $Z^{\mathrm{np}}_{\phi^2}$ given by

$$Z^{\mathrm{np}}_{\phi^2} = 1 - \frac{\mathrm{i}\tilde{c}_{\mathrm{b}}}{\varepsilon}(5^2 3\pi^3 2^2)\left(\frac{A}{k_d g_{\mathrm{R}}^2}\right)^{(d+3)/2} \exp\left(-\frac{A}{k_d g_{\mathrm{R}}^2}\right)(1 + \mathrm{O}(g_{\mathrm{R}}^2)). \tag{2.22}$$

The prescription outlined above successfully renormalises the whole theory to the leading order considered.

3. Renormalisation group functions, fixed points and critical exponents

We can now proceed to calculate the renormalisation group functions which are only strictly correct for $\varepsilon \geqslant 0$. However, in order to obtain a fixed point we have to analytically continue all these functions to $\varepsilon < 0$. The validity of this continuation is one of the main assumptions of the present approach.

The β function is given by

$$\beta(g_{\mathrm{R}}) = (\mu \, \mathrm{d}g_{\mathrm{R}}/\mathrm{d}\mu)_{g_0 \text{ fixed}} \tag{3.1}$$

and so using (2.7), (2.10) and (2.11) we have that

$$\beta(g_{\mathrm{R}}) = \beta_{\mathrm{p}}(g_{\mathrm{R}}) - \mathrm{i}\tilde{c}_{\mathrm{b}}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3\left(\frac{A}{k_d g_{\mathrm{R}}^2}\right)^{(d+4)/2} \exp\left(-\frac{A}{k_d g_{\mathrm{R}}^2}\right)(1 + \mathrm{O}(g_{\mathrm{R}}^2)) \tag{3.2}$$

where β_{p} is the perturbative β function, namely

$$\beta_{\mathrm{p}}(g_{\mathrm{R}}) = -\tfrac{1}{2}\varepsilon g_{\mathrm{R}} - \tfrac{3}{4}k_d g_{\mathrm{R}}^3 + \mathrm{O}(g_{\mathrm{R}}^5). \tag{3.3}$$

The fixed points of the theory are the zeros of (3.2) which also satisfy the constraints imposed by (2.10). Hence we are looking for the solutions of

$$\varepsilon/2 = -\tfrac{3}{4}k_d g_{\mathrm{R}}^2 + \mathrm{O}(g_{\mathrm{R}}^4) - \frac{\mathrm{i}\tilde{c}_{\mathrm{b}} k_d^{1/2}}{A^{1/2}}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3$$
$$\times\left(\frac{A}{k_d g_{\mathrm{R}}^2}\right)^{(d+5)/2} \exp\left(\frac{A}{k_d g_{\mathrm{R}}^2}\right)(1 + \mathrm{O}(g_{\mathrm{R}}^2)) \tag{3.4}$$

subject to the condition Re $g_{\mathrm{R}}^2 > 0$. It turns out that there is a suitable fixed point for $\arg \varepsilon = -\pi$. To calculate this fixed point we need the value of the two-loop perturbative fixed point (de Alcantara Bonfim *et al* 1981). Using this, our fixed point becomes

$$\operatorname*{Re}_{\arg \varepsilon = -\pi} k_d g_{\mathrm{R}}^{*2} = -2\varepsilon/3 - \varepsilon^2 5^3/3^5 + \mathrm{O}(\varepsilon^3) \tag{3.5}$$

$$\operatorname*{Im}_{\arg \varepsilon = -\pi} k_d g_{\mathrm{R}}^{*2} = -B\left(-\frac{18}{5\varepsilon}\right)^{(d+5)/2} \exp\left(\frac{18}{5\varepsilon}\right)(1 + \mathrm{O}(\varepsilon)) \tag{3.6}$$

where

$$B = 5^2(2\pi)^{-9/10}\pi^{-1/2}2^{13/2} \exp\left(\frac{11\zeta'(2)}{2\pi^2} - \frac{3\zeta'(4)}{2\pi^4} - \frac{9\gamma}{2} - \frac{103}{16}\right). \tag{3.7}$$

The corrections-to-scaling exponent ω is defined by

$$\omega = \mathrm{d}\beta(g_{\mathrm{R}})/\mathrm{d}g_{\mathrm{R}}|_{g_{\mathrm{R}}^*}. \tag{3.8}$$

From (3.2)

$$\beta'(g_R) = -\varepsilon/2 + O(g_R^2) - \frac{2i\tilde{c}_b k_d^{1/2}}{A^{1/2}}(5^{1/2}3^{1/2}\pi^{3/2}2^2)^3$$

$$\times\left(\frac{A}{k_d g_R^2}\right)^{(d+7)/2} \exp\left(-\frac{A}{k_d g_R^2}\right)(1+O(g_R^2)) \tag{3.9}$$

and therefore

$$\operatorname*{Im}_{\arg\varepsilon=-\pi} \omega = -(3B/2)\left(-\frac{18}{5\varepsilon}\right)^{(d+7)/2} \exp\left(\frac{18}{5\varepsilon}\right)(1+O(\varepsilon)). \tag{3.10}$$

The renormalisation group function $\gamma_\phi(g_R)$ is defined by

$$\gamma_\phi(g_R) = (1/Z_\phi)(\mu\, dZ_\phi/d\mu)_{g_0 \text{ fixed}} \tag{3.11}$$

where Z_ϕ is the full wavefunction renormalisation, i.e. $Z_\phi = Z_\phi^p Z_\phi^{np}$. Evaluating (3.11), we have that

$$\gamma_\phi(g_R) = \tfrac{1}{6}k_d g_R^2 + O(g_R^4) - i\tilde{c}_b(5^{1/2}3^{1/2}\pi^{3/2}2^2)^2\left(\frac{A}{k_d g_R^2}\right)^{(d+3)/2} \exp -\left(\frac{A}{k_d g_R^2}\right)$$

$$\times(\gamma - \tfrac{1}{2} - \ln 2 + \tfrac{7}{12})(1+O(g_R^2)). \tag{3.12}$$

This function is related to the critical exponent η by $\eta = \gamma_\phi(g_R^*)$. Substituting (3.5) and (3.6) into (3.12), we obtain

$$\operatorname*{Im}_{\arg\varepsilon=-\pi} \eta = -(B/6)\left(-\frac{18}{5\varepsilon}\right)^{(d+5)/2} \exp\left(\frac{18}{5\varepsilon}\right)(1+O(\varepsilon)). \tag{3.13}$$

Finally we can calculate the remaining renormalisation group function, namely

$$\gamma_{\phi^2}(g_R) = (1/Z_{\phi^2})(\mu\, dZ_{\phi^2}/d\mu)_{g_0 \text{ fixed}} \tag{3.14}$$

where $Z_{\phi^2} = Z_{\phi^2}^{n} Z_{\phi^2}^{np}$. The formula for $\gamma_{\phi^2}(g_R)$ is

$$\gamma_{\phi^2}(g_R) = k_d g_R^2 + O(g_R^4) + i\tilde{c}_b(5^2 3\pi^3 2^2)\left(\frac{A}{k_d g_R^2}\right)^{(d+5)/2} \exp\left(-\frac{A}{k_d g_R^2}\right)(1+O(g_R^2)). \tag{3.15}$$

At the fixed point $\gamma_{\phi^2}(g_R^*) = \nu^{-1} - 2 + \eta$ and so

$$\operatorname*{Im}_{\arg\varepsilon=-\pi} \nu^{-1} - 2 + \eta = -(B/4)\left(-\frac{18}{5\varepsilon}\right)^{(d+5)/2} \exp\left(\frac{18}{5\varepsilon}\right)(1+O(\varepsilon)). \tag{3.16}$$

If we had carried out the renormalisation prescription starting with the bare vertex functions for $\arg g_0 = -0$ then we would have obtained a fixed point for $\arg \varepsilon = \pi$. At this fixed point the imaginary parts of ω, η and $\nu^{-1} - 2 + \eta$ have exactly the same form as (3.10), (3.13) and (3.16) respectively, apart from the fact that there is no overall minus sign. The real part is unaffected.

4. High-order behaviour of the critical exponents

The real part of η, ω and $\nu^{-1} - 2 + \eta$ is the same for $\arg \varepsilon = \pi$ and $\arg \varepsilon = -\pi$, while the imaginary part changes sign. We can therefore write a dispersion relation in ε to obtain the high-order behaviour of these expressions for $\varepsilon > 0$. The dispersion relation is

$$F(\varepsilon)=\frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\varepsilon' \frac{\operatorname{Im} F(\varepsilon')_{\arg \varepsilon=\pi}}{\varepsilon'-\varepsilon}. \tag{4.1}$$

In order to evaluate the asymptotic behaviour of the coefficient of ε^k in $F(\varepsilon)$ the integration on the right-hand side of (4.1) is performed by the method of saddle point integration. This leads to the following results:

$$\eta=\sum_k (-1)^{k-1}\frac{B}{6\pi}\varepsilon^k (5/18)^k k!\, k^{9/2}\left(1+\mathrm{O}\left(\frac{\ln k}{k}\right)\right), \tag{4.2}$$

$$\omega=\sum_k (-1)^{k-1}\frac{3B}{2\pi}\varepsilon^k (5/18)^k k!\, k^{11/2}\left(1+\mathrm{O}\left(\frac{\ln k}{k}\right)\right), \tag{4.3}$$

$$\nu^{-1}-2+\eta=\sum_k (-1)^{k-1}\frac{B}{4\pi}\varepsilon^k (5/18)^k k!\, k^{9/2}\left(1+\mathrm{O}\left(\frac{\ln k}{k}\right)\right), \tag{4.4}$$

where B is given by (3.7). The numerical value of B is 0.0172....

5. Conclusions

In the approach previously employed to obtain the high-order behaviour (e.g. McKane 1979), after renormalising, a dispersion relation in g^2 was performed to yield expressions for the β function and for the renormalised vertex functions. From these expressions the various values of a and b were inferred. Here the dispersion relation in g^2 has been replaced by one in ε, and as a consequence of this we have managed to obtain directly the high-order behaviour of all the critical exponents. The values of a and b are in agreement with previous work. We have also managed to obtain the values of the overall universal coefficients, c, of the factorial growth. It is clear therefore that the new approach has certain advantages.

To check the values of c by studying the perturbation theory would involve carrying out the calculation to fairly high order. This can be seen as follows—there are two equivalent ways of writing the coefficient of ε^k in (4.2), (4.3) and (4.4), both of which can be obtained from the dispersion relation (4.1), namely

$$c(-1)^k a^k \Gamma(k+b+1)(1+\mathrm{O}(\ln k/k)) \tag{5.1}$$

and

$$c(-1)^k a^k k!\, k^b(1+\mathrm{O}(\ln k/k)). \tag{5.2}$$

Hence in order to obtain a reliable estimate of c from perturbation theory we would have to carry out the calculation to at least k_c loops, where $k_c! k_c^b \cong \Gamma(k_c+b+1)$. Our observation then follows from the fact that the values of b for this model are relatively large and from the fact that k_c is a monotonic increasing function of b.

In principle analogous calculations to those presented here could be carried out for Hamiltonians of the form

$$\mathscr{H}=\int \mathrm{d}^d\boldsymbol{x}\,[\tfrac{1}{2}(\nabla\boldsymbol{\phi})^2+\tfrac{1}{2}m^2\boldsymbol{\phi}^2+1/3!\,g_0 d_{ijk}\phi_i\phi_j\phi_k] \tag{5.3}$$

where d_{ijk} is a symmetric third-rank invariant tensor of some symmetry group and the number of field components, n, is finite. However, as yet there are no physical realisations of models of the form (5.3) in which the high-order behaviour of the ε expansion is oscillatory. If the high-order behaviour is non-oscillatory then knowing its exact form cannot be put to any practical use; in particular it cannot be used to obtain accurate values of the critical exponents. For these physically realisable models one would find, starting the calculation with arg $g_0 = \pm 0$, a fixed point for arg $\varepsilon = \mp 0$ and the imaginary parts of ω, η and $\nu^{-1}-2+\eta$ at these fixed points have opposite signs. A dispersion relation similar to (4.1) could then be employed to obtain the high-order behaviours for $\varepsilon < 0$. Analytically continuing these to $\varepsilon > 0$ would give the desired high-order behaviours.

We have extended the work of McKane and Wallace (1983). This is not only a technical exercise—there are fundamental differences between the ϕ^3 and ϕ^4 models. For example, in perturbation theory the wavefunction renormalisation comes in at one loop in the ϕ^3 model. We were therefore interested in seeing what differences there might be in renormalising the non-perturbative part and if indeed it was possible to renormalise it. As we have shown, it is possible to renormalise the theory in an analogous way to the ϕ^4 case, as all the necessary cancellations of momentum dependent poles in the imaginary part of the vertex functions do in fact take place. It remains a challenge to understand the systematics of why these cancellations always occur.

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